THE EFFECT OF EXTERNAL DAMPING ON THE STABILITY OF BECK'S COLUMN[1]

by

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ABSTRACT

The stability of a cantilevered column subjected to a constant follower load in the presence of external damping is investigated. It is shown that the buckling load increases with increasing damping from the value $20.05\ EI/L^2$ at zero damping to the limiting value $37.7\ EI/L^2$ for infinite damping.

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The influence of damping on the stability of nonconservative elastic systems has been a subject of recent interest (see [1] for references). The presence of damping has a "stabilizing" effect in some cases and a "destabilizing" effect in others. This note considers the case of Beck's column, a linear elastic cantilevered column subjected to a constant follower load at its free end. External damping of a linear viscous nature is assumed to be present. An approximate expression for the critical load of the column will be derived, and it will be seen that an increase in the damping coefficient causes an increase in the critical load. Contrary to what one might expect, however, the critical load does not become unbounded for large damping.

In nondimensional form, the equation of motion for the lateral displacement $w(x,t)$ of the column is assumed to be

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} + p\,\frac{\partial^2 w}{\partial x^2} + 2\xi\,\frac{\partial w}{\partial t} = 0, \quad 0 < x < 1, \ t \geq 0. \tag{1}$$

Here the x-axis lies along the straight equilibrium shape of the column with $x = 0$ at the built-in end and $x = 1$ at the free end. The time is denoted by t, and $\xi (> 0)$ is the coefficient of damping. The compressive force p is applied at the free end in such a manner that it remains tangential to the column during motion. The boundary conditions are given by

$$w(0,t) = \frac{\partial w}{\partial x}(0,t) = \frac{\partial^2 w}{\partial x^2}(1,t) = \frac{\partial^3 w}{\partial x^3}(1,t) = 0, \quad t \geq 0, \tag{2}$$

while the initial displacement $w(x,0)$ and initial velocity $\frac{\partial w}{\partial t}(x,0)$ are allowed to be arbitrary.

If one considers a solution of the form

$$w(x,t) = a(t)f(x), \tag{3}$$

then (1) separates into the two equations

$$\frac{d^2a}{dt^2} + 2\xi \frac{da}{dt} + \lambda a = 0 \tag{4}$$

and

$$\frac{d^4f}{dx^4} + p\frac{d^2f}{dx^2} - \lambda f = 0 \tag{5}$$

where λ is a constant. Nontrivial solutions of (5) which satisfy the boundary conditions (2) are given by

$$f(x) = \cosh \gamma x - \cos \eta x + \mu(\gamma \sin \eta x - \eta \sinh \gamma x) \tag{6}$$

where

$$\mu = (\gamma^2 \cosh \gamma + \eta^2 \cos \eta)/(\eta\gamma^2 \sinh \gamma + \eta^2 \gamma \sin \eta), \tag{7}$$

$$\gamma = \left\{ -\frac{p}{2} + \sqrt{\lambda + \frac{p^2}{4}} \right\}^{1/2}, \tag{8}$$

$$\eta = \left\{ \frac{p}{2} + \sqrt{\lambda + \frac{p^2}{4}} \right\}^{1/2}, \tag{9}$$

and λ satisfies the frequency equation

$$2\lambda \cos \eta \cosh \gamma + p \quad \lambda \sin \eta \sinh \gamma + 2\lambda + p^2 = 0. \tag{10}$$

As p increases in value from zero, the two lowest roots λ of (10) approach each other. According to Beck [2] these two roots merge when $p = 20.05$ (slightly different values are iven in [3] and [4]). For higher values of p these roots form a complex conjugate pair which shall be denoted by

$$\lambda = \alpha \pm i\beta. \tag{11}$$

Substitution of (11) into (10) yields a complex equation, and setting the real and imaginary parts equal to zero then leads to two transcendental equations in α and β. These equations will not be listed here, but it is noted that they are extremely complicated and untractable. Only the following solution for the special case $\alpha = 0$ has been calculated by the author:

$$\alpha = 0, \ \beta = 191 \quad \text{when} \quad p = 37.7. \tag{12}$$

However, one can easily obtain approximate values for α and β. For example, Deineko and Leonov [3] derived the approximate frequency equation

$$\lambda^2 + (13.36p - 53.34\pi^2)\lambda + 12.14p^2 + 25.75\pi^2 p + 65.38\pi^4 = 0 \qquad (13)$$

with the use of the assumption

$$w(x,t) = \phi(t)y_1(x) + \psi(t)y_2(x) \qquad (14)$$

where

$$\begin{aligned} y_1(x) &= 5 - \cos 2\pi x - 4\cos \pi x, \\ y_2(x) &= 28 - \cos\frac{3\pi x}{2} - 27\cos\frac{\pi x}{2}. \end{aligned} \qquad (15)$$

With $\lambda = \alpha \pm i\beta$, solution of (13) yields

$$\begin{aligned} \alpha &= 26.67\pi^2 - 6.68p, \\ \beta &= [-32.48p^2 + 382.05\pi^2 p - 645.97\pi^4]^{1/2} \end{aligned} \qquad (16)$$

when $p > 20.23$.

Now recall the differential equation (4) for $a(t)$, with solution

$$a(t) = c_1 e^{\Omega_1 t} + c_2 e^{\Omega_2 t} \qquad (17)$$

where

$$\Omega_{1,2} = -\xi \pm \sqrt{\xi^2 - \lambda} \ . \tag{18}$$

If $\lambda = \alpha \pm i\beta$, then

$$\Omega_{1,2} = -\xi \pm \sqrt{\xi^2 - \alpha - i\beta} \quad \text{or} \quad \Omega_{1,2} = -\xi \pm \sqrt{\xi^2 - \alpha + i\beta} \ . \tag{19}$$

The column is asymptotically stable (i.e., $w(x,t) \to 0$ as $t \to \infty$) if the real part of each Ω is negative. For Ω of the form in (19), this stability condition is equivalent to the inequality

$$\alpha > \frac{\beta^2}{4\xi^2} \tag{20}$$

(see [5]). The column is unstable (i.e., $w(x,t)$ can become unbounded as $t \to \infty$) if the real part of at least one Ω is positive, that is, if

$$\alpha < \frac{\beta^2}{4\xi^2} \ . \tag{21}$$

The critical load (or buckling load) p_{cr} is defined such that the column is stable if $p < p_{cr}$ and unstable if $p > p_{cr}$. It follows from (20) and (21) that p_{cr} is the minimum solution p of the equation

$$\alpha = \frac{\beta^2}{4\xi^2} \tag{22}$$

where α and β are functions of p. With the use of the approximate values of α and β given by (16), one obtains

$$26.67\pi^2 - 6.68p_{cr} = \frac{1}{4\xi^2}(-32.48p_{cr}^2+382.05\pi^2 p_{cr}-645.97\pi^4). \tag{23}$$

Solving for p_{cr} gives

$$p_{cr} = 5.88\pi^2 + 0.41\xi^2 - \sqrt{14.71\pi^4+1.55\pi^2\xi^2+0.17\xi^4} . \tag{24}$$

This result is depicted in Figure 1.

A comparison of the approximate expression (24) with the exact critical load may be made for the extreme cases of zero damping and infinite damping. For $\xi = 0$ the column is unstable whenever the frequency equation (10) has complex roots, that is, when $p > 20.05$. The expression (24) yields $p_{cr} = 20.23$ for $\xi \to 0$, a value 1% higher than the exact buckling load.

For the case $\xi \to \infty$, it is seen from (22) that the exact buckling load corresponds to a solution $\alpha = 0$ of the frequency equation (10), and from (12) it then follows that this load is given by $p = 37.7$. According to stability condition (24), $p_{cr} \to 39.4$ as $\xi \to \infty$ (see the dashed line in Figure 1), so that the approximate critical load p_{cr} is $4\frac{1}{2}$% higher than the exact value as $\xi \to \infty$.

In conclusion, it is noted again that the presence of linear viscous external damping has a "stabilizing" effect on Beck's column. The buckling load increases monotonically with increasing damping but does not approach infinity for infinite damping. In fact, the lower and upper bounds on the buckling load are, respectively, $p = 20.05$ (corresponding to zero damping) and $p = 37.7$ (corresponding to infinite damping).

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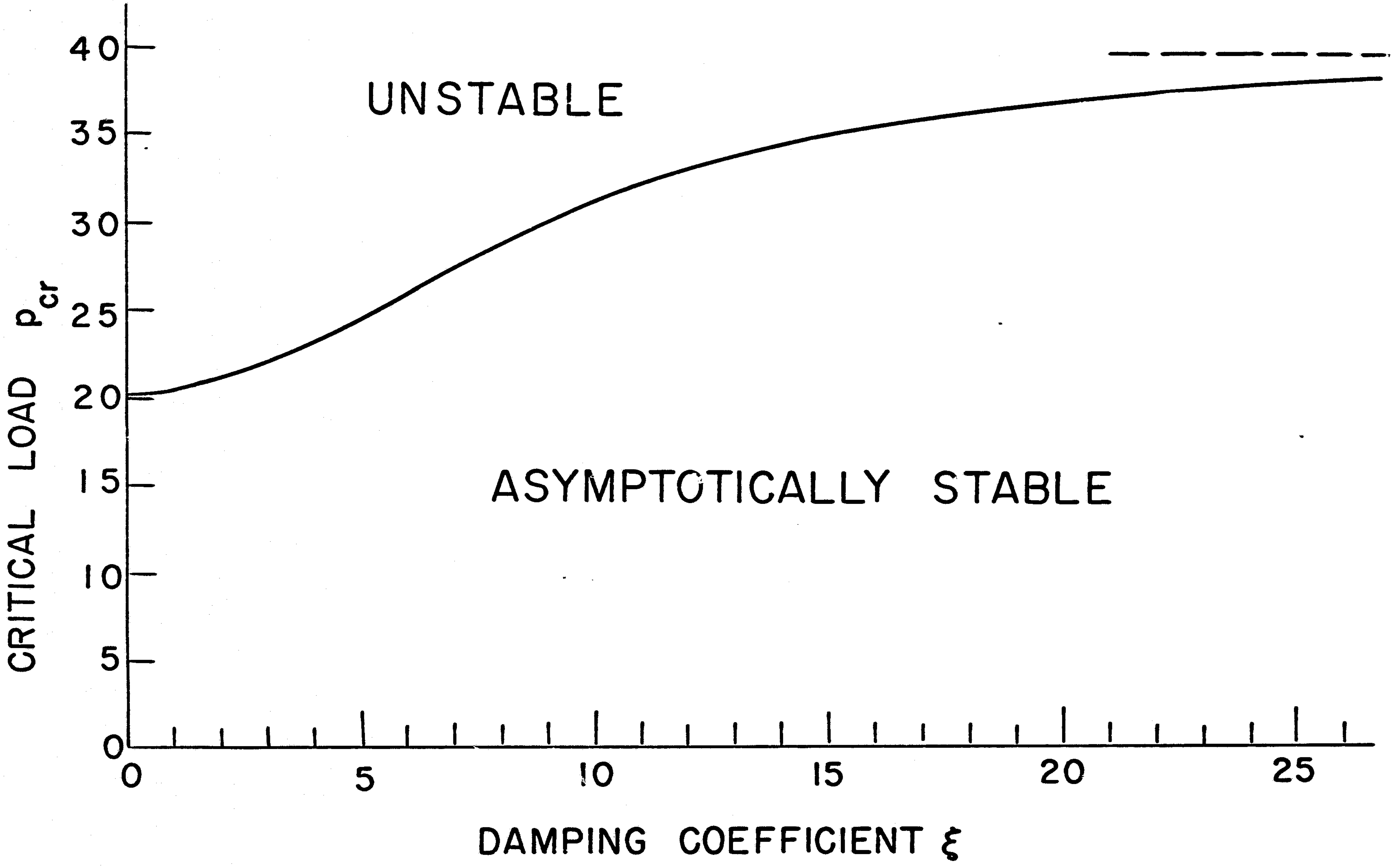
LEGEND FOR FIGURE

Figure 1. Critical load for Beck's column



UNSTABLE
ASYMPTOTICALLY STABLE
CRITICAL LOAD p_{cr}
0
5
10
15
20
25
30
35
40
DAMPING COEFFICIENT ξ